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Quantum Electrodynamics of Particles on a Plane and the Chern-Simons Theory

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Abstract

We study the electrodynamics of generic charged particles (bosons, fermions, relativistic or not) constrained to move on an infinite plane. An effective gauge theory in 2+1 dimensional spacetime which describes the real electromagnetic interaction of this particles is obtained. The relationship between this effective theory with the Chern-Simons theory is explored. It is shown that the QED lagrangian *per se* produces the Chern-Simons constraint relating the current to the effective gauge field in 2+1 D. It is also shown that the geometry of the system unavoidably induces a contribution from the topological θ-term that generates an explicit Chern-Simons term for the effective 2+1 dimensional gauge field as well as a minimal coupling of the matter to it. The possible relation of the effective three dimensional theory with the bosonization of the Dirac fermion field in 2+1 D is briefly discussed as well as the potential applications in Condensed Matter systems.

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1) Introduction

In the last few years there has been an intense theoretical activity focused in the investigation of the properties of charged particles constrained to move on a plane. The main reason for this certainly comes from the recent discoveries of the Quantum Hall Effect [1] and of the High Temperature Superconductivity [2] in the eighties. In both cases the fundamental physical properties of the system seem to be completely determined by the dynamical behavior of a two dimensional array of electrons.

In all of the interesting realistic physical applications, in spite of the fact that the motion of the electrons is confined to a plane, of course the electromagnetic fields through which they interact are certainly not subject to this constraint. The electrostatic potential between two electrons on a plane, for instance, is the familiar 1/r Coulomb potential despite their planar motion and not the logarithmic potential that would be peculiar of electrodynamics in 2+1 dimensions.

Nevertheless, the obtainment from first principles in 3+1 D, of a complete 2+1 dimensional description of this kind of electronic systems which move on a plane but interact, of course, as the four-dimensional particles which they are, would be highly convenient. One would immediately wonder, in particular, what would be the relationship between this effective description and the strict 2+1 dimensional theories that have been constructed in order to explain these systems.

There is a field theory which plays a central role in the set of models built to describe the properties of condensed matter systems in 2+1 D. This is the Chern-Simons theory [3]. In the case of the Hall Effect, it leads very naturally to the transverse conductivity characteristic of this effect. Another important property of the Chern-Simons theory is that it induces a change in the statistics of the particles coupled to it, the so called statistical transmutation [4]. A system of generalized statistics particles or anyons, on the other hand was shown to spontaneously have a superconducting ground state, even in the absence of the attractive phonon interactions which lead to the usual BCS superconductivity [6]. Hence the potential interest in Chern-Simons theories in connection with High-Tc Superconductivity.

The purpose of this work is to obtain an effective 2+1 dimensional field theory describing the real (3+1 D) electromagnetic interaction of generic charged particles (fermionic, bosonic, relativistic or not) constrained to move on an infinite plane. We show that the Maxwell lagrangian of QED leads *per se* to an effective 2+1 D theory which possesses the Chern-Simons constraint, being therefore intimately related to this theory. We also show that the geometry implied by the restriction of the motion to a plane unavoidably induces a contribution from the topological θ-term [7] which will generate the minimal coupling of matter to the effective gauge field in 2+1 D ("statistical field") responsible for the statistical transmutation as well as an explicit Chern-Simons term.

It is important to notice the fundamental difference between the Chern-Simons term which we obtain here which has its origins strictly in the 3+1 dimensional gauge fields and the one which is obtained by integration over the matter fields [8]. The latter, for instance is no longer able to produce a statistical transmutation because after the integration over the matter fields there no longer particles interacting with the Chern-Simons field.

The work is organized as follows. In section 2 we review the main properties of the Chern-Simons theory. In section 3 we obtain the effective electromagnetic interaction for particles constrained to move on a plane. In section 4 we show how to produce this interaction within a 2+1 D field theory. In section 5 we show that the effective gauge field theory in 2+1 D presents the same constraint as the Chern-Simons theory. In section 6 we show how the geometry of the system induces a contribution from the θ-term of QED which will lead to a Chern-Simons explicit term. The conclusions are presented in section 7.

2) The Chern-Simons Theory

The Chern-Simons theory is characterized by the following lagrangian in 2+1 dimensions

$$\mathcal{L} = \frac{\theta}{2}\epsilon^{\mu\alpha\nu}\mathcal{A}_\mu\partial_\alpha\mathcal{A}_\nu - q\ j^\mu\mathcal{A}_\mu + \mathcal{L}_M + \mathcal{L}_{GF} \tag{2.1}$$

where $\mathcal{L}_M$ is the matter lagrangian and $\mathcal{L}_{GF}$ is a gauge fixing term. Also in the above equation, j^μ is the matter current in 2+1 D (which has dimension 2) and q is the (dimensionless) coupling of matter to the gauge field $\mathcal{A}_\mu$ which is usually called the statistical field. θ in (2.1) is an arbitrary real parameter, the "statistical parameter". The field equation associated to (2.1), namely

$$j^\mu = \frac{\theta}{q}\epsilon^{\mu\alpha\nu}\partial_\alpha\mathcal{A}_\nu \tag{2.2}$$

implies that the statistical field is completely determined by the matter current. The field equation is actually a constraint. In the Lorentz gauge, we obtain from (2.2) the expression for $\mathcal{A}_\mu$ in terms of the current

$$\mathcal{A}^\mu = \frac{q}{\theta}\epsilon^{\mu\alpha\nu}\partial_\alpha[\frac{1}{-\Box}]j_\nu \tag{2.3}$$

The constraint (2.2) implies the following relation between the matter current and the electric and magnetic statistical fields

$$j^0 = \frac{\theta}{q}\mathcal{B} \tag{2.4a}$$

$$j^i = \frac{\theta}{q}\epsilon^{ij}\mathcal{E}_j \tag{2.4b}$$

The first of this equations leads to the well known statistical transmutation of the matter fields, the change in statistics being given by $\Delta s = \frac{q^2}{2\pi\theta}$ [4]. This happens because (2.4a) implies that a magnetic flux q/θ is attached to each particle having a charge q [5]. The second one, on the other hand, implies a transverse conductivity which underlies the usefulness of Chern-Simons theories in the description of the Hall effect. We see that the Hall conductance would be given by θ (of course, in any realistic theory of the Hall effect, we should have the real electric field in (2.4b)!).

The Chern-Simons theory produces a peculiar effective long range interaction between matter particles which is, after all, the responsible for the statistical transmutation. We can get this effective interaction by considering the result of the integration over $\mathcal{A}_\mu$ (in euclidean space)

$$Z_{eff}^{CS} = Z^{-1}\int D\mathcal{A}_\mu \exp\{-\int d^3z[-\frac{i\theta}{2}\epsilon^{\mu\alpha\nu}\mathcal{A}_\mu\partial_\alpha\mathcal{A}_\nu + q\, j^\mu\mathcal{A}_\mu + \mathcal{L}_M$$

$$-\frac{\xi\theta}{2}\mathcal{A}^\mu\frac{\partial^\mu\partial^\nu}{(-\Box)^{1/2}}\mathcal{A}_\nu]\} \tag{2.5}$$

The last term in the exponent above is the properly chosen gauge fixing term. Multiplication by $(-\Box)^{-1}$ of course is to be understood in the convolution sense. Integrating over the statistical field with the help of the euclidean propagator of this field, namely

$$G^{\mu\nu} = \frac{i}{\theta}\epsilon^{\mu\alpha\nu}\partial_\alpha[\frac{1}{-\Box_E}] - \frac{1}{\xi\theta}\partial^\mu\partial^\nu[\frac{1}{(-\Box_E)^{3/2}}] \tag{2.6}$$

we get

$$Z^{CS}_{eff} = \exp\{\frac{q^2}{2}\int d^3z_E\ d^3z'_E\ j^\mu(z)G_{\mu\nu}(z-z')j^\nu(z') - \int d^3z\mathcal{L}_M\}$$

$$= \exp\{-\int d^3z_E[\mathcal{L}_M - i\frac{q^2}{2\theta}\epsilon^{\mu\alpha\nu}j_\mu\partial_\alpha[\frac{1}{-\Box_E}]j_\nu]\} \tag{2.7}$$

The last term in the exponent of the above expression is the "statistical interaction", the long range interaction which is responsible for the change in statistics of the matter particles.

3)The Effective Electromagnetic Interaction for Particles on a Plane

Let us consider Quantum Electrodynamics in 3+1 D for generic charged particles

$$\mathcal{L}_{EM} = -\frac{1}{4}F^2_{\mu\nu} - ej^\mu_{3+1}A_\mu + \mathcal{L}_M + \mathcal{L}_{GF} \tag{3.1}$$

In the above expression, $\mathcal{L}_M$ is the matter kinetic lagrangian which is completely arbitrary: it may be relativistic or not and the particles it describes may be either fermions or bosons. $\mathcal{L}_{GF}$ is the gauge fixing term, j^μ_{3+1} is the matter current in 3+1 D and e is the charge of the matter particles.

The electromagnetic field induces an effective interaction of matter which can be obtained by integrating over A_μ in (3.1). Going to euclidean space and using the euclidean propagator of the electromagnetic field, namely

$$G^{\mu\nu}_{EM} = [-\Box_E\delta^{\mu\nu} + (1-\frac{1}{\xi})\partial^\mu\partial^\nu][\frac{1}{(-\Box_E)^2}] \tag{3.2}$$

we get

$$Z^{EM}_{eff} = Z^{-1}\int DA_\mu \exp\{-\int d^4z_E[\frac{1}{4}F^2_{\mu\nu} + e\ j^\mu_{3+1}A_\mu - \frac{\xi}{2}A_\mu\partial^\mu\partial^\nu A_\nu]\}$$

$$= \exp\{\frac{e^2}{2}\int d^4z_E\ d^4z'_E\ j^\mu_{3+1}(z)\ G^{\mu\nu}_{EM}(z-z')\ j^\nu_{3+1}(z')\}$$

$$= \exp\{\frac{e^2}{2}\int d^4z_E\ d^4z'_E\ j^\mu_{3+1}(z)[\frac{1}{-\Box_E}]j^\mu_{3+1}(z')\} \equiv \exp[-S_{eff}[j^\mu_{3+1}]] \tag{3.3}$$

Observe that only the first term of (3.2) contributes to (3.3) due to current conservation. The effective action in (3.3) is the familiar electromagnetic interaction. For static point charges, for instance, the energy corresponding to it is the familiar 1/r Coulomb potential energy.

Since we are interested in describing particles in 2+1 D, i.e., matter confined to a plane, let us consider the case in which the current is given by

$$j^\mu_{3+1}(x^0,x^1,x^2,x^3) = \begin{cases} j^\mu(x^0,x^1,x^2)\delta(x^3) & \mu=0,1,2 \\ 0 & \mu=3 \end{cases} \tag{3.4}$$

Inserting this expression (euclideanized) in the effective action in (3.3) and integrating over z^3 and z'^3 we get

$$S_{eff} = -\frac{e^2}{2}\int d^3z_E\ d^3z'_E\ j^\mu(z)K_E(z-z'|z^3=z'^3=0)j^\mu(z') \tag{3.5}$$

where the euclidean kernel K_E is given by

$$K_E(z-z'|z^3=z'^3=0) \equiv [\frac{1}{(-\Box)_E}]_{z^3=z'^3}$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{\exp[i\sum_{i=1}^4 k\cdot(z_i-z'_i)]}{\sum_{i=1}^4\ k_i^2}\ |_{z_3=z'_3=0} = \frac{1}{8\pi^2\sum_{i=1,2,4}(z_i-z'_i)^2} \equiv \frac{1}{8\pi^2|z-z'|^2_{3D}} \tag{3.6}$$

where we have defined the 3-D vectors $z_{3D}=(z_1,z_2,z_4)$.

Equations (3.5) and (3.6) determine the electromagnetic interaction of matter which is confined to a plane as is implied by the expression for the current, eq.(3.4). We shall see in the next section how this effective interaction can be obtained from a theory in 2+1 dimensional spacetime.

4) The Pseudo Electromagnetic Field

Let us start by observing that expression (3.6) for the effective kernel can be written as a three-dimensional integral

$$\frac{1}{8\pi^2|z-z'|^2_{3D}} = \frac{1}{4}\int \frac{d^3k_{3D}}{(2\pi)^3}\frac{e^{i[k\cdot(z-z')]_{3D}}}{(k^2_{3D})^{1/2}} \equiv \frac{1}{4}\;\frac{1}{(-\Box^E_{2+1})^{1/2}} \tag{4.1a}$$

where $\Box^E_{2+1}$ is the (euclidean) three-dimensional d'Alembertian operator. Replacing (3.6) for (4.1a) and inserting in (3.5) we can see that the effective electromagnetic interaction for the charged particles on a plane is already completely expressed within a three-dimensional world:

$$S_{eff} = -\frac{e^2}{8}\int d^3z_E\; d^3z'_E\; j^\mu(z)[\frac{1}{(-\Box^E_{2+1})^{1/2}}](z-z')j^\mu(z') \tag{4.1b}$$

We will show in this section that we can obtain this effective electromagnetic interaction by starting from a three-dimensional field theory in which the 2+1 D matter current j^μ interacts with the 2+1 D gauge field $\bar{A}_\mu$, which we call the pseudo electromagnetic field. Let us consider the following lagrangian for the $j^\mu - \bar{A}_\mu$ system in 2+1 dimensions (we henceforth drop the subscript "2+1" in the three dimensional d'Alembertian)

$$\mathcal{L}_{PEM} = -\frac{1}{4}\bar{F}_{\mu\nu}[\frac{1}{\Box^{1/2}}]\bar{F}^{\mu\nu} - \frac{e}{2}j^\mu\bar{A}_\mu + \frac{\xi}{2}\bar{A}_\mu\frac{\partial^\mu\partial^\nu}{\Box^{1/2}}\bar{A}_\nu \tag{4.2}$$

where the last term above is a gauge fixing term.

Going to euclidean space and integrating over the $\bar{A}_\mu$ field, with the help of its euclidean propagator, namely

$$D^{\mu\nu} = [-\Box_E\delta^{\mu\nu} + (1-\frac{1}{\xi})\partial^\mu\partial^\nu][\frac{1}{(-\Box_E)^{3/2}}] \tag{4.3}$$

we get

$$Z^{PEM}_{eff} \equiv \exp[-S_{eff}] = \exp\{\frac{e^2}{8}\int d^3z_E\; d^3z'_E\; j^\mu(z)[\frac{1}{(-\Box_E)^{1/2}}]j^\mu(z')\} \tag{4.4}$$

Observe that because of current conservation, only the first term of (4.3) contributes in (4.4). Comparing with (4.1b), we conclude, therefore, that the theory described by

(4.2)(in 2+1 D) reproduces precisely the same effective interaction which was obtained from QED after imposing that the charges are confined to a plane. This fact was shown here in euclidean space. Going to Minkowski space, one can show [9] that the identity (4.1a) is also valid. Choosing a certain prescription (e.g. Feynman, advanced , etc) for the kernel in the $\bar{A}_\mu$-field theory would lead to the corresponding kernels of QED. In particular, it is interesting to note that the theory described by (4.2) in 2+1 D would produce a Coulombic (1/r) interaction between static charges in the plane, which is the correct result for real charges, instead of the logarithmic(unphysical) potential which is known to be produced by three-dimensional QED.

The theory described by (4.2) was proposed to be associated with the bosonization of the free Dirac fermion field in 2+1 D [10]. As we shall comment later, it is rather suggestive that it also appears in the present context.

5) The Statistical Field and the Chern-Simons Constraint

Let us show here that the theory of the pseudo electromagnetic field defined by (4.2) is equivalent to the following theory involving the gauge field $\mathcal{A}_\mu$ which for reasons that will become clear later, will shall call the statistical field. Consider

$$\mathcal{L}[\mathcal{A}_\mu, \bar{A}_\mu] = -\frac{\lambda^2}{4}\mathcal{F}_{\mu\nu}[\frac{1}{\Box^{1/2}}]\mathcal{F}^{\mu\nu} + \lambda\epsilon^{\mu\alpha\beta}\bar{A}_\mu\partial_\alpha\mathcal{A}_\beta + \lambda^2\frac{\xi}{2}\mathcal{A}_\mu\frac{\partial^\mu\partial^\nu}{\Box^{1/2}}\mathcal{A}_\nu \tag{5.1}$$

where the last term is the gauge fixing and λ is an arbitrary real constant.

Going to euclidean space and integrating over the $\mathcal{A}_\mu$ field we get the effective $\bar{A}_\mu$ action:

$$Z_{eff}[\bar{A}_\mu] \equiv \exp\{-S_{eff}[\bar{A}_\mu]\} = Z_0^{-1}\int D\mathcal{A}_\mu \exp\{-\int d^3z[\frac{\lambda^2}{2}\mathcal{A}_\mu[\frac{-\Box_E\delta^{\mu\nu} + (1-\xi)\partial^\mu\partial^\nu}{(-\Box_E)^{1/2}}]\mathcal{A}_\nu$$

$$-i\lambda\epsilon^{\mu\nu\alpha}\bar{A}_\mu\partial_\nu\mathcal{A}_\alpha]\}$$

$$= \exp\{-\frac{\lambda^2}{2}\int d^3z_E\ d^3z'_E\ \bar{A}_\mu(z)\epsilon^{\mu\sigma\alpha}\partial_\sigma\ \bar{A}_\nu(z')\epsilon^{\nu\lambda\beta}\partial'_\lambda[\frac{1}{\lambda^2}D^{\mu\nu}(z-z')]\} \tag{5.2}$$

In the last term, the expression between brackets is the euclidean propagator for the field $\mathcal{A}_\mu$, where $D^{\mu\nu}$ is given by (4.3). Inserting (4.3) in (5.2) we immediately find that only the first term contributes. After a little manipulation and analytic continuation back to the Minkowski space, we conclude that the effective lagrangian for the $\bar{A}_\mu$ field obtained by integration over $\mathcal{A}_\mu$ in (5.1) is

$$\mathcal{L}_{eff}[\bar{A}_\mu] = -\frac{1}{4}\bar{F}_{\mu\nu}\Box^{-1/2}\bar{F}^{\mu\nu} \tag{5.3}$$

This is precisely the first term of the lagrangian (4.2) of the pseudo electromagnetic field which, as we saw, describes correctly the real electromagnetic interaction within a 2+1 D formulation. We therefore are going to rewrite the lagrangian (4.2) in the following way

$$\mathcal{L}_{PEM}[\bar{A}_\mu, \mathcal{A}_\mu] = -\frac{\lambda^2}{4}\mathcal{F}_{\mu\nu}\Box^{-1/2}\mathcal{F}^{\mu\nu} - (\frac{e}{2}j^\mu - \lambda\epsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta)\bar{A}_\mu + \mathcal{L}_{GF} \tag{5.4}$$

As showed above, integration over $\mathcal{A}_\mu$ will produce the lagrangian (4.2). Instead of doing so, however, let us integrate over the $\bar{A}_\mu$ field, in order to get the effective lagrangian for $\mathcal{A}_\mu$. Adding the matter field kinetic lagrangian to (5.4) we have the following vacuum functional (in Minkowski space)

$$Z_{2+1} = Z_0^{-1}\int D\bar{A}_\mu D\mathcal{A}_\mu D\psi \exp\{i\int d^3z[-\frac{\lambda^2}{4}\frac{\mathcal{F}^2_{\mu\nu}}{\Box^{1/2}} - (\frac{e}{2}j^\mu - \lambda\epsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta)\bar{A}_\mu + \mathcal{L}_M + \mathcal{L}_{GF}]\} \tag{5.5}$$

where ψ represents the matter fields.

Integrating over the field $\bar{A}_\mu$ in (5.5), as we promised, we see that we produce a functional delta function identifying the matter current with the topological current of the $\mathcal{A}_\mu$ field, namely

$$Z_{2+1} = Z_0^{-1}\int D\mathcal{A}_\mu D\psi\ \delta[\frac{e}{2}j^\mu - \lambda\epsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta] \exp\{i\int d^3z[-\frac{\lambda^2}{4}\frac{\mathcal{F}^2_{\mu\nu}}{\Box^{1/2}} + \mathcal{L}_M + \mathcal{L}_{GF}]\} \tag{5.6}$$

We see that the constraint generated in the theory which represents the electromagnetic interaction of QED in a three-dimensional formulation is precisely the Chern-Simons constraint (2.2) relating the matter current to the statistical field. From (2.2) we see that the relation between λ and the Chern-Simons parameter θ must be $\theta = 2\lambda$. This explains why we called $\mathcal{A}_\mu$ the "statistical field".

Observe that the only interaction between the matter field and the $\mathcal{A}_\mu$-field is the one induced by the constraint. There is in particular no minimal coupling between these fields. One can see directly from (5.6) that this peculiar interaction does produce the correct electromagnetic interaction for the carged matter. Indeed, writing the effective lagrangian for the $\mathcal{A}_\mu$-matter system as (with a slight abuse of notation for the constraint)

$$
\mathcal{L}[\mathcal{A}_\mu, \psi] = \mathcal{L}_M - \frac{\lambda^2}{4}\frac{\mathcal{F}_{\mu\nu}^2}{\Box^{1/2}} + constraint[j^\mu \equiv \frac{2\lambda}{e}\epsilon^{\mu\alpha\beta}\partial_\alpha \mathcal{A}_\beta]
$$

$$
= \mathcal{L}_M + \frac{\lambda^2}{2}\epsilon^{\mu\alpha\beta}\partial_\alpha \mathcal{A}_\beta[\Box^{-1/2}]\epsilon_{\mu\sigma\lambda}\partial^\sigma \mathcal{A}^\lambda + constraint[j^\mu \equiv \frac{2\lambda}{e}\epsilon^{\mu\alpha\beta}\partial_\alpha \mathcal{A}_\beta] \tag{5.7}
$$

it is easy to see that integration over $\mathcal{A}_\mu$ with the help of the delta function constraint, will produce the following effective interaction for the matter field

$$
\mathcal{L}_{eff}^{EM} = \mathcal{L}_M - \frac{e^2}{8} j^\mu \Box^{-1/2} j_\mu \tag{5.8}
$$

This, according to (4.1b), describes the correct electromagnetic interaction for particles moving on a plane.

Starting from QED in 3+1 D and restricting the motion of the charged particles to a plane we have arrived at an effective theory in 2+1 D which possesses the same constraint as the Chern-Simons theory. We have seen in Section 2 that the statistics transmutation taking place in Chern-Simons theory is a direct consequence of this constraint. One is therefore naturally led to ask whether the theory described by (5.7) should also induce a statistical transmutation on the charged matter particles coupled to $\mathcal{A}_\mu$. The answer for this question is no. The reason is that there is no minimal coupling between the matter current j^μ and the $\mathcal{A}_\mu$-field in (5.7). In order to have statistical transmutation we need both the constraint (2.2) and a minimal coupling to the matter current.

A simple way to prove the previous statement is to consider a static point charge minimally coupled to the statistical field. Using the constraint, eq. (2.4), we see that there is going to be a point statistical magnetic flux associated to this static point charge. This configuration of the statistical magnetic field corresponds to a vector potential $\mathcal{A}_i = \partial_i arg(\vec{x})$. If there is a minimal coupling between the current and the vector potential $\mathcal{A}_\mu$ we immediately realize that we can eliminate this configuration by performing a gauge transformation on the matter fields. As a consequence, these fields will acquire a phase proportional to $arg(\vec{x})$. It is clear that the transformed fields will get an extra phase under rotations of 2π after the transformation. In other words it will have its spin/statistics changed. In the absence of the minimal coupling, as in (5.7), on the other hand, the presence of the point magnetic flux would not imply a change in statistics through the above gauge transformation.

Another way of seeing that the lagrangian (5.7) will not produce a change in the statistics of the matter fields is to observe that the effective interaction in (5.8) does not contain a term like the one appearing in (2.6) which is the effective interaction responsible for the change in statistics.

In the next section we are going to see how an explicit Chern-Simons term for the $\mathcal{A}_\mu$-field, as well as a $j^\mu - \mathcal{A}_\mu$ coupling, will be induced starting from 3+1 D.

6) Induction of the Chern-Simons Term and Statistical Transmutation

Let us start by considering the topological θ-term action for the electromagnetic field, namely

$$S_\theta = -\frac{\theta}{4} \int d^4x F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{6.1}$$

where θ is an arbitrary real number. This action is usually uninteresting both at classical and quantum level. It is the integral of a total derivative

$$S_\theta = -\frac{\theta}{2} \int d^4x \; \partial_\mu I^\mu \qquad ; \quad I^\mu = \epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \tag{6.2}$$

and therefore it does not affect the equations of motion and bears no effect on the classical behavior of the system. At the quantum level, on the other hand, the introduction of (6.1) in the fuctional integral would in principle have consequences on the quantum behavior of the system. It happens however that S_θ is a topological invariant which classifies the homotopy classes of the Π_3 mappings and for an abelian (U(1)) field like A_μ this mapping is always trivial. (This of course is not the case for nonabelian fields where the θ term is known to lead to the nontrivial vacuum structure of the theory [11]). We are going to see, however, that when we constrain the charged particles to move on an infinite plane substrate as we have been doing in the previous sections the S_θ term (6.1) does produce a nontrivial effect on the dynamics of the effective 2+1 dimensional system. Indeed, considering the geometry depicted in Fig.1 which is appropriate for the system we have been investigating and applying the Gauss theorem to (6.2), we get

$$S_\theta = -\frac{\theta}{2}[\int_{z_3=0} d^3\xi^\mu I_\mu + \int_{S_\infty} d^3\xi^\mu I_\mu] \tag{6.3}$$

Neglecting the term involving the surface at infinity, noting that for the first term in (6.3) (for the surface at $z^3 = 0$) $d^3\xi^\mu = -d^3\xi^3 = -dx^0 dx^1 dx^2$ and using the conventions $\epsilon^{0123} = \epsilon_{3012} = 1$ and $\epsilon^{012} = \epsilon_{012} = 1$ we immediately see that we obtain the Chern-Simons action

$$S_\theta = \frac{\theta}{2}\int d^3x \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta \tag{6.4}$$

This fact was exploited in [7] where a four dimensional representation of the Chern-Simons action was used.

We see that starting in 3+1 D from the action

$$S_\theta + S_I = \int d^4x \; [-\frac{\theta}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} - e \; j^\mu_{3+1} A_\mu] \tag{6.5}$$

we arrive in 2+1 D at

$$S = \int d^3x [\frac{\theta}{2} \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta - e \; j^\mu A_\mu] \tag{6.6}$$

This will induce the effective interaction which we found in (2.7) and which is responsible for the statistical transmutation.

Going back to the lagrangian (4.2) which was obtained from QED by dimensional reduction we see, on the basis of the above reasoning, that when we constrain the system to the geometry of Fig. 1, we must also take into account the θ-term for the electromagnetic field. We therefore arrive at the following effective lagrangian in 2+1 D

$$\mathcal{L}_{PEM}[\bar{A}_\mu, A_\mu] = -\frac{1}{4} \bar{F}_{\mu\nu} (\Box^{-1/2}) \bar{F}^{\mu\nu} - \frac{e}{2} j^\mu \bar{A}_\mu - e j^\mu A_\mu + \frac{\theta}{2} \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta \tag{6.7}$$

In this equation A_μ is the θ-induced 2+1 D relic of the electromagnetic field.

Going to the $\mathcal{A}_\mu$-field language and integrating over $\bar{A}_\mu$ as we did in section 5, we will immediately obtain

$$\mathcal{L}[\mathcal{A}_\mu, A_\mu] = -\frac{\lambda^2}{4}\frac{\mathcal{F}^2_{\mu\nu}}{\Box^{1/2}} + \mathcal{L}_M + \frac{\theta}{2}\epsilon^{\mu\alpha\beta}A_\mu\partial_\alpha A_\beta - ej^\mu A_\mu + constraint[j^\mu \equiv \frac{2\lambda}{e}\epsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta] \tag{6.8}$$

Substituting j^μ for the constraint in (6.8), going to euclidean space and integrating over A_μ with the help of the euclidean propagator (2.6) we generate the following θ-dependent term for $\mathcal{A}_\mu$

$$\begin{aligned} Z_\theta[\mathcal{A}_\mu] &= Z_0^{-1}\int DA_\mu \exp\{-\int d^3z_E[-\frac{i\theta}{2}\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu + 2i\lambda\epsilon^{\mu\alpha\beta}A_\mu\partial_\alpha\mathcal{A}_\beta] \\ &= \exp\{-2\lambda^2\int d^3z\ d^3z'\ \mathcal{A}_\mu(z)\epsilon^{\mu\sigma\alpha}\partial_\sigma\mathcal{A}_\nu(z')\epsilon^{\nu\lambda\beta}\partial'_\lambda[\frac{i}{\theta}\epsilon^{\alpha\rho\beta}\partial_\rho[\frac{1}{-\Box_E}]]\} \\ &= \exp\{-i\frac{2\lambda^2}{\theta}\int d^3z_E\ \epsilon^{\mu\alpha\nu}\mathcal{A}_\mu\partial_\alpha\mathcal{A}_\nu\} \end{aligned} \tag{6.9}$$

This expression is telling us that the θ-term generates a Chern-Simons term for the statistical field $\mathcal{A}_\mu$ as well. Going to Minkowski space and adding the result of the integration (6.9) to (6.8), we get the complete $\mathcal{A}_\mu$ lagrangian

$$\mathcal{L}[\mathcal{A}_\mu] = -\frac{\lambda^2}{4}\frac{\mathcal{F}^2_{\mu\nu}}{\Box^{1/2}} - \frac{2\lambda^2}{\theta}\epsilon^{\mu\alpha\nu}\mathcal{A}_\mu\partial_\alpha\mathcal{A}_\nu + \mathcal{L}_M + constraint[j^\mu \equiv \frac{2\lambda}{e}\epsilon^{\mu\alpha\beta}\partial_\alpha\beta] \tag{6.10}$$

Making the choice $\lambda = \frac{\theta}{2}$, as we did in section 5, and using the constraint we may finally write

$$\mathcal{L}[\mathcal{A}_\mu] = -\frac{1}{4}\mathcal{F}_{\mu\nu}[\frac{\theta^2}{4\Box^{1/2}}]\mathcal{F}^{\mu\nu} + \frac{\theta}{2}\epsilon^{\mu\alpha\nu}\mathcal{A}_\mu\partial_\alpha\mathcal{A}_\nu - ej^\mu\mathcal{A}_\mu + \mathcal{L}_M + constraint[j^\mu \equiv \frac{\theta}{e}\epsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta] \tag{6.11}$$

This is our final expression for the complete Chern-Simons theory including the true elctromagnetic interaction between charged particles constrained to move on an infinite plane. Using the constraint on the first term in (6.11) will immediately produce the electromagnetic interaction (4.1b) or (3.5). Insertion of the constraint in the second and third terms of (6.11), on the other hand, and using the corresponding solution for $\mathcal{A}_\mu$ in terms of j^μ, Eq. (2.3), will produce the statistical interaction which we have in (2.7). Indeed, integration over the $\mathcal{A}_\mu$-field with the help of the delta function constraint yields the following effective lagrangian for the charged matter

$$\mathcal{L}_M^{eff} = \mathcal{L}_M - \frac{e^2}{8} j^\mu \frac{1}{\Box^{1/2}} j_\mu + \frac{e^2}{2\theta} \epsilon^{\mu\alpha\beta} j_\mu \partial_\alpha [\frac{1}{\Box}] j_\beta \tag{6.12}$$

The first term is the kinetic lagrangian and is completely general.The second term is the electromagnetic interaction and the third term is the satatistical interaction induced by the θ-term in the 2+1 D space.

The presence of the minimal coupling along with the constraint in (6.11) indicate that by the gauge transformation described at the end of section 5 the charged fields will suffer a change $\Delta S = \frac{e^2}{2\pi\theta}$ in their spin/statistics.

Apart from reproducing the true electromagnetic interaction within the framework of 2+1 dimensional space-time (and in particular the 1/r Coulomb potential between static point carges) the theory described by (6.11) has a number of very interesting features. The $\mathcal{A}_\mu$-field part of the lagrangian (6.11) is precisely what one obtains as the bosonic field lagrangian in the bosonization of the three-dimensional Dirac fermion field in 2+1 D [10]. It is quite suggestive that it appears here in a different context but also associated with a change in statistics. The full consequences of this connection where certainly not yet completely explored and would be worthwile to understand more profoundly.

The quantization of theories possesing the nonlocality of the type appearing in (6.11) was studied in detail in [9]. A nice property of (6.11) is that the propagator of the $\mathcal{A}_\mu$-field (for retarded or advanced prescriptions) has support on the light-cone surface as in the case of photons in 3+1 D and therefore the theory obeys the Huygens principle in the same way as four-dimensional QED but unlike its three-dimensional counterpart [9, 12]. This observation allows us to conclude that in spite of the nonlocality of the first term in (6.11) the theory does respect causality, because by choosing retarded or advanced prescriptions, for instance, one can show that the kernel $\Box^{-1/2}$, the same appearing in the $\mathcal{A}_\mu$ propagator, has support on the light cone surface [9, 12].

7) Conclusions

We have seen that the electrodynamics of particles moving on an infinite plane leads to an effective 2+1 dimensional theory which possesses the Chern-Simons constraint relating the matter current to the gauge field. The geometry of the system, on the other hand, implies that the contribution of the topological θ-term can be no longer neglected. An explicit Chern-Simons term for the 2+1 dimensional gauge field as well as a coupling of the matter to it are then induced in the effective 2+1 dimensional theory. It is important to remark that the Chern-Simons induction we consider here is completely different from the one which is produced by integration over the matter fields [8]. The Chern-Simons term we found here coexists with the matter fields.

The effective theory we obtained completely describes the real electromagnetic interaction of the charged particles, in spite of being three dimensional and provides therefore a very convenient framework for the description of realistic Condensed Matter systems like the electron gas undergoing the Quantum Hall Effect or maybe the High-Tc Superconductors. It would be extremely interesting to investigate the behavior of the theory introduced here in the presence of an external electromagnetic field. In the case of a constant external magnetic field, for instance, one would have the situation relevant for the Quantum Hall Effect. It is probable that the magnetic field would tune the statistical parameter θ in such a way as to produce the observed effects in the conductance, for instance. We are presently investigating this question.

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Figure Caption

Fig.1 Geometry of the system of charged particles moving on an infinite plane and the surface used in the Gauss theorem (dashed line).